IMO Number Theory Questions

Level: Intermediate Ref No: M06 Puzz Points: 13

[Cayley 2004 Q6] How many different solutions are there to this word sum, where each letter stands for a different non-zero digit?

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   M A T H S
 + M A T H S
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 C A Y L E Y
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Solution: 2 possibilities, 74269 + 73269 = 148538 and 74329 + 74329 = 148658

Level: Intermediate Ref No: M07 Puzz Points: 15

[Hamilton 2004 Q1]
(a) A positive integer N is written using only the digits 2 and 3, with each appearing at least once. If N is divisible by 2 and by 3, what is the smallest possible integer N?
(b) A positive integer M is written using only the digits 8 and 9, with each appearing at least once. If M is divisible by 8 and by 9, what is the smallest possible integer M?

Solution: (a) 2232 (b) 8 888 889 888

Level: Intermediate Ref No: M20 Puzz Points: 10

[Cayley 2006 Q2] Show that there are no solutions to this "letter sum". [Each letter stands for one of the digits 0-9; different letters stand for different digits; no number begins with the digit 0.]

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 S E V E N
 +   O N E
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 E I G H T
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Level: Intermediate Ref No: M22 Puzz Points: 10

[Cayley 2006 Q4] Find the positive integer whose value is increased by 518059 when the digit 5 is placed at each end of the number.

Solution: 2006

Level: Intermediate Ref No: M25 Puzz Points: 15

[Hamilton 2006 Q1] Find the smallest positive integer which consists only of 0s and 1s when written in base 10, and which is divisible by 12. (*Base 10 just means our standard number system; specifically it means that each place value has ten possible digits, i.e. 0 to 9*)

Solution: 11100

Level: Intermediate Ref No: M31 Puzz Points: 20

[Maclaurin 2006 Q1] Find the smallest positive multiple of 35 whose digits are all the same as each other.

Solution: 555555

Level: Intermediate Ref No: M42 Puzz Points: 13

[Cayley 2007 Q6] You are told that one of the integers in a list of distinct positive integers is 97 and that their average value is 47.
(a) If the sum of all the integers in the list is 329, what is the largest possible value for a number in the list?
(b) Suppose the sum of all the numbers in the list can take any value. What would the largest possible number in the list be then?

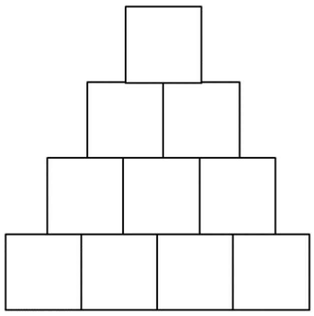
Solution: (a) 217 (b) 1078

Level: Intermediate   Ref No: M43   Puzz Points: 15

[Hamilton 2007 Q1] Numbers are placed in the blocks shown alongside according to the following two rules.

(a) For two adjacent blocks in the bottom row, the number in the block to the right is twice the number in the block to the left.
(b) The number in the block above the bottom row is the sum of the numbers in the two adjacent blocks immediately below it.

What is the smallest positive integer that can be placed in the bottom left-hand block so that the sum of all ten numbers is a cube?



Solution: 300

Level: Intermediate   Ref No: M44   Puzz Points: 15

[Hamilton 2007 Q2] Prove that there is exactly one sequence of five consecutive positive integers in which the sum of the squares of the first three integers is equal to the sum of the squares of the other two integers.

Solution: 10, 11, 12, 13, 14

Level: Intermediate   Ref No: M55   Puzz Points: 10

[Cayley 2011 Q1] A palindromic number is one which reads the same when its digits are reversed, for example 23832. What is the largest six-digit palindromic number which is exactly divisible by 15?

Solution: 597795

Level: Intermediate Ref No: M59 Puzz Points: 13

[Cayley 2011 Q5] Solve the equation $5a - ab = 9b^2$, where a and b are positive **integers**.

Solution: $a = 12, b = 2$ and $a = 144, b = 4$

Level: Intermediate Ref No: M63 Puzz Points: 15

[Hamilton 2011 Q3] A particular four-digit number *N* is such that:
a) The sum of *N* and 74 is a square; and
b) The difference between *N* and 15 is also a square.
What is the number *N*?

Solution: 1951

Level: Intermediate Ref No: M67 Puzz Points: 20

[Maclaurin 2011 Q1] How many positive integers have a remainder of 31 when divided into 2011?

Solution: 21

Level: Intermediate Ref No: M70 Puzz Points: 20

[Maclaurin 2011 Q4] How many solutions are there to the equation $x^2 + y^2 = x^3$, where x and y are positive integers and x is less than 2011?

Solution: 44

Level: Intermediate Ref No: M73 Puzz Points: 10

[Cayley 2008 Q1] How many four-digit multiples of 9 consist of four different odd digits?

Solution: 24

Level: Intermediate Ref No: M76 Puzz Points: 10

[Cayley 2008 Q4] The number *N* is the product of the first 99 positive integers. The number *M* is the product of the first 99 positive integers after each has been reversed. That is for example, the reverse of 8 is 8; of 17 is 71; and of 20 is 02.

Find the exact value of $N \div M$.

Solution: 10^9

Level: Intermediate Ref No: M84 Puzz Points: 20

[Maclaurin 2008 Q1] All the digits of a certain positive three-digit number are non-zero. When the digits are taken in reverse order a different number is formed. The difference between the two numbers is divisible by eight.

Given that the original number is a square number, find its possible values.

Solution: 169, 961

Level: Intermediate Ref No: M86 Puzz Points: 20

[Maclaurin 2008 Q3] Show that the equation:

$$\frac{1}{x}+\frac{1}{y}=\frac{5}{11}$$

Has no solutions for positive integers x, y.

Solution: $(5x-11)(5y-11)=121$ has no solutions because all possible factor pairs of 121 lead to non-integer values for either x or y.

Level: Intermediate Ref No: M96 Puzz Points: 15

[Hamilton 2009 Q2] Find the possible values of the digits p and q, given that the five-digit number 'p543q' is a multiple of 36.

Solution: $p=4, q=2$ and $p=9, q=6$

Level: Intermediate Ref No: M102 Puzz Points: 20

[Maclaurin 2009 Q2] Miko always remembers his four-digit PIN (personal identification number) because
a) It is a perfect square, and
b) It has the property that, when it is divided by 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, there is always a remainder of 1.
What is Miko's PIN?

Solution: 5041

Level: Intermediate Ref No: M105 Puzz Points: 23

[Maclaurin 2009 Q5] A lottery involves five balls being selected from a drum. Each ball has a different positive integer printed on it.
Show that, whichever five balls are selected, it is always possible to choose three of them so that the sum of the numbers on these three balls is a multiple of 3.

Solution: Remainders of each ball divided by 3 are 0, 1 or 2. If three of the balls have same remainder, sum will be exactly divisible by 3. Otherwise there will be at least one ball of each remainder. Since $0+1+2=3$, choosing one ball of each remainder yields a multiple of 3.

Level: Intermediate Ref No: M107 Puzz Points: 10

[Cayley 2010 Q1] The sum of three positive integers is 11 and the sum of the cubes of these numbers is 251. Find all such triples of numbers.

Solution: 2, 3, 6 and 1, 5, 5

Level: Intermediate Ref No: M109 Puzz Points: 10

[Cayley 2010 Q3] Find all possible solutions to the ‘word sum’ on the right.

Each letter stands for one of the digits 0-9 and has the same meaning each time it occurs. Different letters stand for different digits. No number starts with a 0.

$$\begin{array}{cccc} & \mathrm{O} & \mathrm{D} & \mathrm{D} \\ + & \mathrm{O} & \mathrm{D} & \mathrm{D} \\ \hline \mathrm{E} & \mathrm{V} & \mathrm{E} & \mathrm{N} \end{array}$$

Solution: Two possibilities only: 655 + 655 = 1310, 855 + 855 = 1710

Level: Intermediate Ref No: M112 Puzz Points: 13

[Cayley 2010 Q6] A ‘qprime’ number is a positive integer which is the product of exactly two different primes, that is, one of the form $q \times p$, where q and p are prime and $q \neq p$.
What is the length of the longest possible sequence of *consecutive* integers all of which are qprime numbers?

Solution: 3

Level: Intermediate Ref No: M127 Puzz Points: 10

[Cayley 2005 Q4] The five-digit number ‘$a679b$’, where a and b are digits, is divisible by 36. Find all possible such five-digit numbers.

Solution: 36792 and 86796

Level: Intermediate Ref No: M133 Puzz Points: 15

[Hamilton 2005 Q4] An ‘unfortunate’ number is a positive integer which is equal to 13 times the sum of its digits. Find all ‘unfortunate’ numbers.

Solution: 117, 156, 195

Level: Intermediate Ref No: M136 Puzz Points: 20

[Maclaurin 2005 Q1] From a three-digit number (with no repeated digit and no zero digit) we can form six two-digit numbers by choosing all possible ordered pairs of digits. For example, the number 257 produces the six numbers 25, 52, 57, 75, 27, 72.
Find all such three-digit numbers with no repeated digit for which the sum of the six two-digit numbers is equal to the original three-digit number.

Solution: 132, 264, 396

Level: Intermediate Ref No: M139 Puzz Points: 20

[Maclaurin 2005 Q4] Consider the following three equations:

$$11 - 2 = 3^2$$
$$1111 - 22 = 33^2$$
$$111111 - 222 = 333^2$$

Prove that the pattern suggested by these three equations continues for ever.

Solution: Note that 111..11 with $2k$ digits can be expressed as $\left(10^{2k} - 1\right) \div 9$. And similarly 222…2 with k digits can be expressed as $2\left(10^k - 1\right) \div 9$. Subtracting these algebraic fractions gives $\left(\frac{10^k-1}{3}\right)^2$, which we can see is the form of the square of k digit 3s in the same way.